

Polygon dissections and Euler, Fuss, Kirkman and Cayley numbers

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Abstract

We give a short and elementary proof of the formulas for classical numbers of polygon dissections. We describe the relationship between the proof, our recent work in knot theory, and Jones' work on planar algebras.

Over one hundred years ago Arthur Cayley considered the following problem:

“ The partitions are made by non-intersecting diagonals; the problem which have been successively considered are

- (1) to find the number of partitions of an r -gon into triangles,*
- (2) to find the number of partitions of an r -gon into k parts, and*
- (3) to find the number of partitions of an r -gon into p -gons, r of the form $n(p-2)+2$.*

Problem (1) is a particular case of (2); and it is also a particular case of (3); but the problem (2) and (3) are outside each other;... ” [Cayley, On the partition of a polygon, March 12, 1891, [Ca]]

Our work on knot theory motivated an elementary and short “bijective” proof of the common generalization of (2) and (3).

Let $Q_i(s, n)$ denote the set of dissections of a convex $(sn+2)$ -gon by i non-crossing diagonals into $sj+2$ -gons ($1 \leq j \leq n-1$), i.e. we allow dissections which can be subdivided to dissections into $(s+2)$ -gons. Let $q_i(s, n) = |Q_i(s, n)|$ be the cardinality of $Q_i(s, n)$. For $i = n-1$ we get a solution to Problem (3) – the Fuss¹ number, [Fu]. For $s = 1$ we get a solution to Problem (2) – the Kirkman-Cayley number, [Ki, Ca]. Compare [Be, S1]. In particular for $i = n-1$ and $s = 1$ we get a solution to Problem (1) – the Euler-Catalan

¹ When Leonard Euler, after an eye operation in 1772, became almost completely blind, he asked Daniel Bernoulli in Basel to send a young assistant, well trained in mathematics, to him in St. Petersburg. It was Nikolaus Fuss who accepted this plea; he arrived in Petersburg in May 1773; [Oz].

number². The reader may find various combinatorial interpretations of the Catalan numbers in [S2].

We find the formula for $q_i(s, n)$ by studying dissections with an additional structure – a specified region of the dissected polygon. Let us denote by $P_i(s, n)$ the set of all dissections in $Q_i(s, n)$ with such a specified region. Note that i diagonals always dissect a polygon into $i + 1$ regions and hence the number of elements of $P_i(s, n)$ is $p_i(s, n) = (i + 1)q_i(s, n)$.

Theorem 0.1 *There is a bijection $\Psi_{s,n}$ between the dissections in $P_i(s, n)$ and the Cartesian product of the set of all sequences $1 \leq a_1 \leq a_2 \leq \dots \leq a_i \leq sn + 2$ and the set of all sequences $(\epsilon_1, \epsilon_2, \dots, \epsilon_{n-1}) \in \{0, 1\}^{n-1}$ with exactly i epsilons equal to 1.*

Since the number of sequences $1 \leq a_1 \leq a_2 \leq \dots \leq a_i \leq sn + 2$ is $\binom{sn+2+i-1}{i}$ and the number of epsilon sequences is $\binom{n-1}{i}$ we get as a corollary the number of dissections in $P_i(s, n)$ and in $Q_i(s, n)$.

Corollary 0.2 $q_i(s, n) = \frac{1}{i+1}p_i(s, n) = \frac{1}{i+1}\binom{sn+i+1}{i}\binom{n-1}{i}$.

Note that the presence of a specified base region in any dissection D in $P_i(s, n)$ induces an orientation on all diagonals in D – every diagonal is oriented in such a way that if you travel along it according to its orientation, then the base region is on the left. In particular every diagonal has a beginning.

Let $v_1, v_2, \dots, v_{sn+2}$ denote the vertices of a convex polygon ordered cyclically in the anti-clockwise direction. For any dissection D in $P_i(s, n)$ the sequence $1 \leq a_1 \leq a_2 \leq \dots \leq a_i \leq sn + 2$ associated with D by Ψ_n is just the sequence of all indices of vertices which are the beginnings of diagonals in D . The sequence $(\epsilon_1, \epsilon_2, \dots, \epsilon_{n-1})$ has a less intuitive meaning and it will be defined by the induction in n in the course of the proof of Theorem 0.1.

Proof by induction on n :

- (1) If $n = 1$ then $i = 0$, and $P_i(s, n)$ has only one (empty) dissection. Since there is only one sequence $a_1 \leq a_2 \leq \dots \leq a_i$ (the empty sequence) and there is only one sequence $(\epsilon_1, \epsilon_2, \dots, \epsilon_{n-1}) \in \{0, 1\}^{n-1}$ (the empty sequence), clearly $\Psi_{s,1}$ is a (uniquely defined) bijection.

$(n \Rightarrow n + 1)$ Assume that we already constructed a bijection

$$\Psi_{s,n} : P_i(s, n) \rightarrow \{1 \leq a_1 \leq \dots \leq a_i \leq sn + 2\} \times \{(\epsilon_1, \epsilon_2, \dots, \epsilon_{n-1}), i \text{ epsilons equal } 1\}$$

for every s and i , $i \leq n - 1$. We are going to construct a bijection $\Psi_{s,n+1}$. Consider a dissection D in $P_i(s, n + 1)$ with the sequence of the beginnings of its diagonals

² Euler posed this problem in 1751 to Christian Goldbach. Euler himself said: “the process of induction I employed was quite laborious” [Do].

$v_{a_1}, v_{a_2}, \dots, v_{a_i}$, $i \leq n$. Consider the first j such that $v_{a_{j+1}}$ is at least $s+1$ vertices apart in the anti-clockwise direction from v_{a_j} . By Dirichlet (Pigeon hole) argument such j exists. We set $\epsilon_1 = 1$ if $v_{a_j}v_{a_{j+s+1}}$ is an edge of the dissection and $\epsilon_1 = 0$ otherwise³. Hence $a_j + s + 1$ is always well defined.

Now, by the inductive assumption, we assign to the truncated $(sn+2)$ -gon $v_1, v_2, \dots, v_{a_j}, v_{a_j+s+1}, \dots, v_{s(n+1)+2}$ dissected by $(i-\epsilon_1)$ diagonals a sequence $(\epsilon_2, \epsilon_3, \dots, \epsilon_n)$ with $(i-\epsilon_1)$ 1's. Note that the base region is never cut off from the original $s(n+1)+2$ -gon. Therefore the resulted dissected polygon is an element of $P_{i-\epsilon_1}(s, n)$ and by the inductive assumption there is a sequence of $n-1$ epsilons assigned to it, $(\epsilon_2, \epsilon_3, \dots, \epsilon_n)$ with $i-\epsilon_1$ epsilons equal to 1. In this way we have assigned to the original dissection D the sequences $a_1 \leq a_2 \leq \dots \leq a_i$, $(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ and hence we have defined $\Psi_{s,n+1}$.

In order to see that $\Psi_{s,n+1}$ is a bijection we construct the inverse function to it. Consider any sequence $1 \leq a_1 \leq a_2 \leq \dots \leq a_i \leq s(n+1)+2$, and any sequence $(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ with i epsilons equal to 1. We construct a corresponding dissection D in $P_i(s, n+1)$ as follows. Consider the first j such that $v_{a_{j+1}}$ is at least $s+1$ vertices apart in the anti-clockwise direction from v_{a_j} . Then $v_{a_j}v_{a_{j+s+1}}$ is a diagonal in D if $\epsilon_1 = 1$ and it is not a diagonal in D otherwise. All other diagonals of D are the diagonals of the dissection $D' \in P_{i-\epsilon_1}(s, n)$ of an $sn+2$ -gon $v_1, v_2, \dots, v_{a_j}, v_{a_j+s+1}, \dots, v_{s(n+1)+2}$, with the sequence of $i-1$ vertices $v_{a_1}, v_{a_2}, \dots, v_{a_{j-1}}, v_{a_{j+1}}, \dots, v_{a_i}$, if $\epsilon_1 = 1$, or with the sequence of i vertices $v_{a_1}, v_{a_2}, \dots, v_{a_{j-1}}, v_{a_j}, v_{a_{j+1}}, \dots, v_{a_i}$ if $\epsilon_1 = 0$. The sequence of epsilons associated with D' is $(\epsilon_2, \dots, \epsilon_n)$. The dissection D' exists by the inductive assumption (which ensures that $\Psi_{s,n}$ is a bijection.) The specified region of D is the specified region of D' .

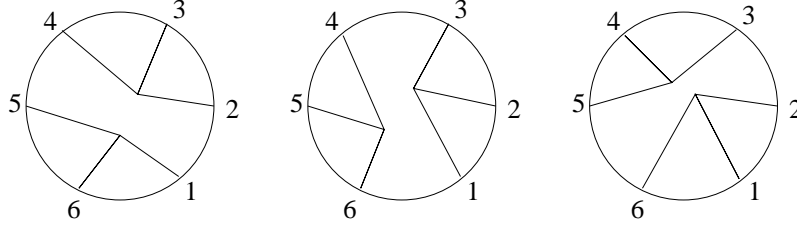
□

The Fuss number can be given yet two additional combinatorial interpretations. The first interpretation goes back probably to the Euler-Fuss tradition (see [Y-Y]) and the second was recently introduced by Bisch and Jones, [B-J].

Consider a disc D^2 with sn vertices on its boundary, v_1, v_2, \dots, v_{sn} . An s -spider is a tree in D^2 composed of one s -valent vertex lying inside the disc and of s edges connecting it with some vertices $v_{i_1}, v_{i_2}, \dots, v_{i_s}$. We denote by $F(s, n)$ the set of all possible combinations of n

³We consider the indices of vertices of the $(s(n+1)+2)$ -gon modulo $s(n+1)+2$.

non-intersecting s -spiders in D^2 . For example $F(3, 2)$ has three elements presented below.



Theorem 0.3 *The number of elements of $F(s, n)$ is equal to $f(s, n) = \frac{1}{(s-1)n+1} \binom{sn}{n}$.*

In this interpretation of Fuss numbers it is convenient to consider s shifted by 1. Note that $f(s, n) = \frac{1}{n(s-1)+1} \binom{sn}{n} = \frac{1}{n} \binom{sn}{n-1} = q_{n-1}(s-1, n)$ is indeed a Fuss number.

Proof: For the sake of the proof of Theorem 0.3 it is convenient to consider spiders in an annulus $A = D^2 \setminus D_0^2$, $D_0^2 \subset \text{int} D^2$, with sn vertices v_1, v_2, \dots, v_{sn} on its external boundary. We denote by $A(s, n)$ the set of all possible combinations of n non-intersecting spiders in A . Note that one can distinguish in each spider S in A its first leg in the following way. There is a point $p \in \partial D^2$ which can be connected with ∂D_0 by a curve which does not intersect the spider S . The first leg of S is the first vertex v_i in S in the anti-clockwise direction from p .

For each collection of n spiders in A we can assign a sequence of their first legs, $v_{i_1}, v_{i_2}, \dots, v_{i_n}$. We claim that this assignment leads to a bijection between elements of $A(s, n)$ and the set of all sequences $1 \leq i_1 < \dots < i_n \leq sn$. This claim follows from the following lemma.

Lemma 0.4 *For any sequence $1 \leq i_1 < \dots < i_n \leq sn$ there is a unique collection of n non-intersecting spiders in A whose first legs are $v_{i_1}, v_{i_2}, \dots, v_{i_n}$.*

Proof by induction on n : For $n = 1$ the above statement is obvious. Suppose the lemma is true for n . Let $v_{i_1}, v_{i_2}, \dots, v_{i_{n+1}}$ be a sequence of vertices chosen from $s(n+1)$ vertices lying on the boundary of D^2 . Let v_{i_j} be the first vertex in this sequence such that $v_{i_{j+1}}$ is at least s vertices apart in the anti-clockwise direction from v_{i_j} ⁴. Note that if $v_{i_1}, v_{i_2}, \dots, v_{i_{n+1}}$ are the first legs of a collection of non-intersecting spiders in D^2 then one of the spiders in it, call it S , has its feet in $v_{i_j}, v_{i_j+1}, \dots, v_{i_j+s-1}$. We cut out a region $W \subset A$ from A which contains S and which is adjacent to the external boundary of A in such a way that $A \setminus W$ is also an annulus with the vertices $v_1, \dots, v_{i_j-1}, v_{i_j+s}, v_{i_j+1}, \dots, v_{s(n+1)}$ on its boundary. By the inductive assumption there is a unique collection of n spiders in $A \setminus W$ whose first legs are $v_{i_1}, v_{i_2}, \dots, v_{i_j-1}, v_{i_j+1}, \dots, v_{i_{n+1}}$. These spiders together with S form the required collection of $n+1$ spiders in A . \square

⁴ i_{j+1} is considered modulo $s(n+1)$.

By the above lemma $A(s, n)$ has $\binom{sn}{n}$ elements. Since $A \subset D^2$ we have an obvious function $A(s, n) \rightarrow F(s, n)$. Note that n spiders always dissect D^2 into $(s-1)n+1$ regions and therefore the above function is $(s-1)n+1$ to 1. This implies that $|F(s, n)| = \frac{1}{(s-1)n+1} \binom{sn}{n}$. \square

We consider now the Bisch-Jones interpretation of the Fuss numbers. Consider a disk D with $2(s-1)n$ vertices on its boundary divided into $2n$ blocks of $s-1$ vertices. The blocks are enumerated (in the anti-clockwise direction) from 1 to $2n$. Furthermore, vertices in each block are labeled (colored) by integers $1, 2, \dots, s-1$ or by integers $s-1, \dots, 2, 1$, depending on whether the number of the block was odd or even.

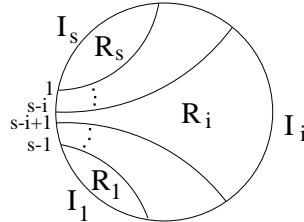
One is interested in the number of complete pairings of vertices in D . By a complete pairing we mean a collection of disjoint arcs in D connecting pairs of vertices of the same color such that every vertex is an end of exactly one arc in D . Denote the set of such connections by $F'(s, n)$.

Theorem 0.5 *There is a bijection $\Phi_{s,n}$ between the sets $F'(s, n)$ and $F(s, n)$.*

In particular, $|F'(s, n)| = \frac{1}{(s-1)n+1} \binom{sn}{n}$, compare [B-J].

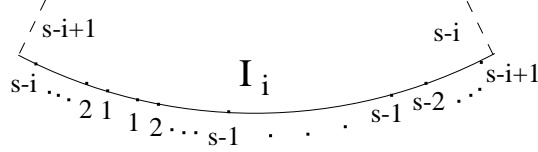
Proof by induction on n : Since $F(s, 1)$ and $F'(s, 1)$ have both a single element, the bijection $\Phi_{s,1} : F'(s, 1) \rightarrow F(s, 1)$ is obvious.

Assume now that we have already defined a bijection $\Phi_{s,n}$ for any s and any $n < N$. We are going to construct a bijection $\Phi_{s,N} : F'(s, N) \rightarrow F(s, N)$ for any given s i.e. we will assign to any disc D with a complete pairing of its $2N(s-1)$ points, $D \in F'(s, N)$ a collection of N s -spiders in a disc D' with sN enumerated vertices. Consider the arcs in D which join the vertices of the first block (labeled by $1, 2, \dots, s-1$) with other vertices of D . They divide D into regions R_1, R_2, \dots, R_s , each of which is a topological disc. R_1 is bounded by the arc labeled by $s-1$ and a segment of ∂D . R_i , $2 \leq i \leq s-1$, is bounded by arcs labeled by $s-i+1$ and $s-i$, a small segment of ∂D lying between the $(s-i+1)$ -th and $(s-i)$ -th vertex of the first block and by another segment, I_i , of ∂D .



We want to study R_i on its own as a disc with a complete pairing of its vertices. Therefore we consider all the vertices of D lying on ∂R_i as vertices of R_i except the four

vertices being the ends of the two arcs $R_i \cap \partial D$ ⁵. If R_i contains some vertices then they lie in I_i and they have a form presented below.

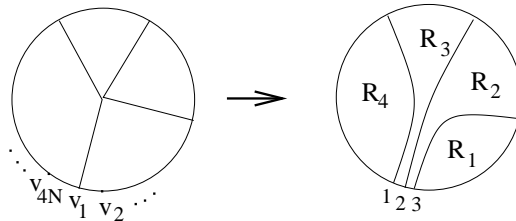


Join the first $s - i$ vertices and the last $i - 1$ vertices of I_i into one block in ∂R_i and enumerate all blocks of vertices in ∂R_i in the anti-clock-wise manner starting from the first full block in I_i with its vertices labeled by increasing numbers. Note that this enumeration assigns odd (respectively: even) numbers to blocks with their vertices labeled by increasing (respectively: decreasing) numbers. Therefore $R_i \in F'(s, r_i)$, where $2r_i$ is the number of blocks in the boundary of R_i .

Consider now a disc D' with vertices v_1, v_2, \dots, v_{sN} , and an s -spider whose one leg is in v_1 and all other legs are chosen in such a way that the spider separates D' into regions R'_1, \dots, R'_s , with R'_i having exactly sr_i vertices on its boundary, for $i = 1, 2, \dots, s$. By the inductive assumption $\Phi_{s, 2r_i}(R_i)$ is a collection of r_i s -spiders in a disc with sr_i enumerated vertices. We paste these spiders into R'_i region. There is an ambiguity in doing this which we resolve by demanding that the first vertex of $\Phi_{s, 2r_i}(R_i)$ goes to the i -th vertex of R'_i , i.e if $v_a, v_{a+1}, \dots, v_{a+sr_i-1}$ are vertices of R'_i then the first vertex of the disc $\Phi_{s, 2r_i}(R_i)$ goes to v_{a+i-1} . The obtained collection of $\sum_{i=1}^s r_i = N$ spiders in D' is the element of $F(s, N)$ associated to $D \in F'(s, N)$ by $\Phi_{s, N}$.

Lemma 0.6 $\Phi_{s, N}$ is a bijection.

Proof: We show this by constructing $\Phi_{s, N}^{-1} : F(s, N) \rightarrow F'(s, N)$. Let $D \in F(s, N)$ be a collection of N s -spiders in a disc with sN vertices. Consider the spider in D one of whose legs is in v_1 . Replace the vertex v_1 by a block of $s - 1$ vertices labeled by $1, 2, \dots, s - 1$ and replace the spider by $s - 1$ disjoint curves as shown below in the example for $n = 4$.



These curves separate D into s regions R_1, R_2, \dots, R_s having sr_1, sr_2, \dots, sr_n vertices respectively. We define $I_i \subset \partial R_i \cap \partial D, i = 1, 2, \dots, s$, as before. Note that $r_i < N$ and therefore, by the inductive assumption Φ_{s, r_i} is a bijection for $i = 1, 2, \dots, s$. If $r_1 > 0$ then

⁵ $R_i \cap \partial D$ is only a single arc for $i = 1$ and $i = s$. Therefore we ignore only two vertices in these cases.

$\Phi_{s,r_1}^{-1}(R_1)$ is a disc with $2r_1$ enumerated blocks of vertices on its boundary which has its vertices paired by sr_i disjoint arcs respecting the labeling of the vertices. We replace R_1 by $\Phi_{s,r_1}^{-1}(R_1)$ in such a way that the first vertex of the first block of $\Phi_{s,r_1}^{-1}(R_1)$ is the s -th vertex on the arc I_1 . If $r_1 = 0$ then we leave R_1 unchanged. Similarly, if $r_2 > 0$ then we replace R_2 by $\Phi_{s,r_2}^{-1}(R_2)$ in such a way that the first vertex of the first block in $\Phi_{s,r_2}^{-1}(R_2)$ is the $(s-1)$ -th vertex on I_2 . We proceed analogously for $i = 3, \dots, s$. After completing this process we receive a disc with $2N$ blocks of vertices, where the vertices of every odd (resp: even) block are labeled by $1, 2, \dots, s-1$, (resp: $s-1, \dots, 2, 1$). Since all vertices are paired by non-intersecting curves respecting their labeling, the disc which we obtained represents an element of $F'(s, N)$. It is not difficult to check that $\Phi_{s,N}^{-1}(D') = D$. Therefore the proofs of Lemma 0.6 and Theorem 0.5 are completed. \square

Remark 1 The proofs given in this paper were inspired by the first author's work on (relative) skein modules which appear in knot theory. These are modules (over a ring) associated with surfaces and 3-dimensional manifolds, [P1, P2]. In particular, the relative skein module of a disc has a basis composed of all pairings of $2n$ points on the boundary of a disc. Similarly, the relative skein module of the annulus has a basis composed of all pairings of $2n$ points lying on the external boundary of an annulus.

Remark 2 We can generalize Theorem 0.3 by considering all possible i element combinations of s -spiders in a disc D with sn vertices, $i \leq n$, which can be completed to a full n element collection of non-intersecting s -spiders in D . We denote the number of such combinations by $d_i(s, n)$. Note that $d_i(s, n)$ is a generalization of Fuss numbers analogous to $q_i(s, n)$. Similarly we can consider all i element combinations of s -spiders in an annulus with sn vertices on its external boundary, which can be completed to n element combinations of spiders. We denote the number of such combinations by $a_i(s, n)$.

By considering ϵ sequences (as in the proof of Theorem 0.1) one can prove the following generalization of Theorem 0.3.

Theorem 0.7 (i) $a_i(s, n) = \binom{sn}{i} \binom{n}{i}$

$$(ii) \ d_i(s, n) = \frac{1}{i(s-1)+1} \binom{sn}{i} \binom{n}{i}$$

Note that although both $q_i(s, n)$ and $d_i(s, n)$ generalize Fuss numbers they agree only when they are Fuss numbers,

$$d_n(s, n) = \frac{1}{n(s-1)+1} \binom{sn}{n} = \frac{1}{n} \binom{sn}{n-1} = q_{n-1}(s-1, n).$$

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